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# SOME NEW THEORETICAL RESULTS FOR ORTHOTROPIC INFLATABLE BEAMS

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## ABSTRACT

A so-called inflatable beam is a membrane tube which gains its stiffness only by a prescribed internal pressure. Such lightweight structures are useful in inflatable buildings and particularly in space industry, whenever rapidly deployable and easily transportable structures are required.

An inflatable beam may look like a standard beam, but its mechanism is quite different and a specific mechanical study is necessary in order to establish its governing equations. The only way to correctly take account of the internal pressure – which is a follower load – is to carry out a complete nonlinear analysis before any possible linearization.

This work aims to extend the analytical results for isotropic inflatable beams to the practical situation when the beam is made of an orthotropic fabric. It will be shown that the Timoshenko beam model combined with the finite rotation kinematics enables one to correctly account for all the nonlinear terms in the governing equations. A linearization will then be performed to study deformations about the pre-stressed reference configuration. Furthermore, it will be shown how to precisely compute the radius and the length of the inflated tube in the reference state, a very significant issue which is overlooked so far in the literature. All the proposed analytical results will be compared with the numerical ones obtained from a nonlinear 3D finite element code.

## NOMENCLATURE

### *Coordinates systems*

- (**x,y,z**) fixed orthonormal Cartesian basis
- (*X,Y,Z*) reference Cartesian co-ordinates
- (*r, φ, X*) cylindrical co-ordinates
- (*ℓ, t, n*) warp, weft, normal directions of the fabric
- (**e<sub>ℓ</sub>, e<sub>r</sub>, e<sub>t</sub>**) the orthonormal local basis with vectors directed along the warp, normal and weft directions

### *Beam geometry*

- $\ell_\circ$  natural length of the inflatable beam
- $R_\circ$  natural radius
- $h_\circ$  natural thickness of the membrane
- $\ell_0$  reference length of the inflatable beam
- $R_0$  reference radius
- $h_0$  reference thickness of the membrane
- $S_0$  reference cross section area of the inflatable beam
- $I_0$  reference second moment of area of the cross section

### *Kinematics*

- U** displacements field
- (*U, V*) axial displacement along **x** and deflection along **y**
- R** rotation tensor
- $\theta$  rotation around **z**-axis
- E** Green strain tensor

### Internal forces

- $\Pi$  first Piola-Kirchhoff stress tensor
- $\Sigma$  second Piola-Kirchhoff stress tensor
- $\Sigma^0$  pre-stress tensor (existing in the reference configuration)
- $N$  nominal axial force
- $T$  nominal shear force along  $y$ -axis
- $M$  nominal bending moment around  $z$ -axis

### Loads

- $p$  internal pressure
- $P = p\pi R_0^2$  pressure resultant
- $N^0$  axial force due to the internal pressure

### Mechanical properties

- $E_\ell, E_t$  modulus of elasticity in weft and warp directions of the orthotropic fabric
- $G_{\ell t}$  in-plane shear modulus
- $\nu_{\ell t}, \nu_{t\ell}$  Poisson ratios

## INTRODUCTION

Inflatable structures are more and more used nowadays due to their interesting properties. They are light, easy to fold, easy to transport, easy to deploy and they are not too expensive to manufacture. The space industry launches inflatable structures such as deployable antennas or inflatable re-entry capsules. On Earth, more and more inflatable buildings are developed, which can be used for example in temporary events.

Inflatable structures are often made of assemblies of elementary parts like tubes, arches or cones. Each part is a membrane-like envelope made of an inducted airtight fabric submitted to an internal pressure which ensures a pre-stress in the fabric and allows the inflatable structure to support external loads such as the self-weight, the snow weight or the wind effects.

When studying inflatable structures, one has to distinguish three states: (i) the first one is the *natural* state where there is no pressure in the structure and the geometry and material properties are known. (ii) the second one is the *initial* or *reference* state where the beam is subjected to the internal pressure. This is the configuration about which the linearized equations have to be obtained. (iii) and eventually, the *deformed* state is obtained when other external loadings are applied.

Strength of materials theory was commonly used to predict the behavior of the structure between the initial and the deformed state. It was mainly developed for homogeneous isotropic fabrics. The first results on the deflection and collapse load of cantilever inflatable tubes can be found in Comer and Levy's work [1], where the inflatable beams were modeled as standard Euler-Bernoulli beams. Main et al. [2] made experiments on a cantilever isotropic beam and improved Comer's theory. More

recently, Suhey et al. [3] reconsidered the theory in [2] and incorporated the finite element into Main's model.

Other works were carried out later with the aim of improving the previous formulations. It was found that Euler-Bernoulli beam theory is not satisfactory since the tube cross sections do not remain orthogonal to the neutral fibre of the tube during deformation and most importantly the inflation pressure did not appear in the expression for the deflection.

Fichter [4] improved the previous theories by using Timoshenko's kinematics and minimizing the total potential energy. This was a seminal paper which greatly influenced many of the subsequent works on inflatable structures. Wielgosz and Thomas [5] [6] derived analytical solutions for inflated panels and tubes by using the Timoshenko's kinematics and by writing the equilibrium equations in the deformed state in order to take into account the internal pressure. Le van and Wielgosz [7] improved Fichter's theory by applying the 2D Timoshenko's kinematics and the Lagrangian form of the virtual work principle with finite displacements and rotations, and derived the nonlinear equations for inflatable isotropic beams. Apedo et al. [8] went further using a 3D Timoshenko's kinematics and derived their solutions for a homogeneous orthotropic fabric in finite displacements and small rotations.

This paper is devoted to the analytical formulation of inflatable beams made of an orthotropic fabric. Use will be made of the total Lagrangian formulation so as to correctly take account of the highly pressurized initial state of the beam and the change in geometry under the internal pressure which is a follower force. The Timoshenko's kinematics is chosen to describe finite rotations. Linearizing the problem around the pre-stressed state eventually leads to a set of equations which clearly shows the influence of the internal pressure on the bending and shear stiffnesses. Furthermore, exact relations for the radius and the length of inflated tube in the pre-stressed state will be provided, these are very significant results which are overlooked so far in the literature. All the proposed analytical results will be compared with the numerical ones obtained from a nonlinear 3D finite element code coping with membrane structures.

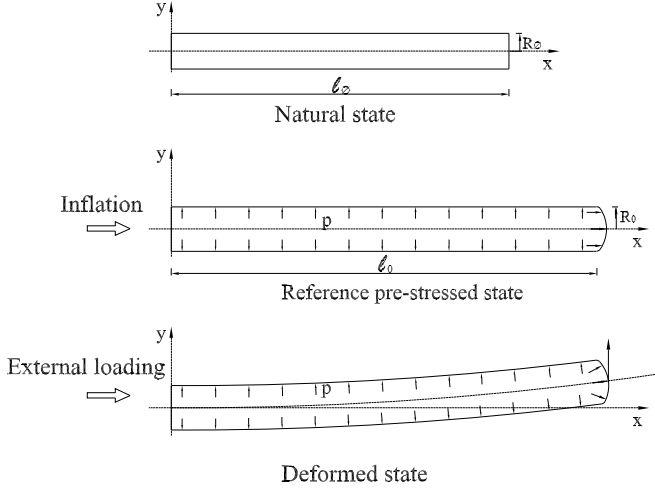
## 1 DEFINITION OF THE ORTHOTROPIC INFLATABLE BEAM PROBLEM

In this work we consider the inflation and the bending of an inflatable beam made of an orthotropic fabric. The beam is an airtight membrane tube which is cylindrical in the natural state, of thickness  $h_0$ , radius  $R_0$  and length  $\ell_0$ , Figure 1a. The specific feature of such a structure is that it has no stiffness in the natural state and the only way for it to bear external loads is to subject the beam to an internal pressure.

In the theory of inflated beams, it turns out to be more convenient to take the pre-stressed state where the beam is inflated by an internal pressure  $p$  – rather than the natural state – as the

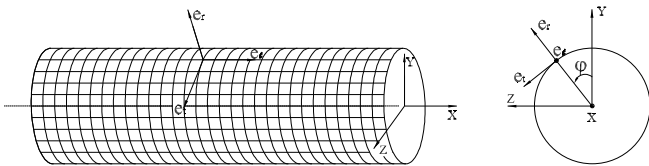
reference state, Figure 1b. In order to distinguish between the natural and reference states, use will be made of index  $\circ$  for the natural state and index 0 for the reference one throughout the paper. Thus, we denote by  $h_0, R_0, \ell_0, S_0$  and  $I_0$ , the thickness, the radius, the length, the cross section area and the second moment of area of the beam in the reference state, respectively.

Eventually, other external loads can be applied to cause bending of the beam as shown in Figure 1c.



**FIGURE 1.** NATURAL, REFERENCE AND DEFORMED STATES OF THE INFLATED BEAM

The beam is assumed to be made of an orthotropic fabric, with the warp direction parallel to the beam axis. As shown in Figure 2, the fixed orthonormal Cartesian basis identical to the beam axes is denoted  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  and the orthonormal local basis directed along the warp, normal and weft directions is denoted  $(\mathbf{e}_\ell = \mathbf{x}, \mathbf{e}_r, \mathbf{e}_t)$ . One has  $\mathbf{e}_r = \cos \varphi \mathbf{y} + \sin \varphi \mathbf{z}$ ,  $\mathbf{e}_t = -\sin \varphi \mathbf{y} + \cos \varphi \mathbf{z}$ , where  $\varphi$  is the angle between  $\mathbf{y}$  and  $\mathbf{e}_r$ ,



**FIGURE 2.** LOCAL AND FIXED BASES

## 2 NONLINEAR EQUATIONS SET

The governing equations will be derived from the Lagrangian form of the virtual power principle: for all virtual velocity field  $\mathbf{V}^*$ ,

$$-\int_{\Omega_0} (\mathbf{F}\boldsymbol{\Sigma})^T : \mathbf{grad} \mathbf{V}^* d\Omega_0 + \int_{\Omega_0} \rho_0 f_0 \mathbf{V}^* d\Omega_0 + \int_{\partial\Omega_0} \mathbf{T} \mathbf{V}^* dS_0 = 0 \quad (1)$$

where  $\Omega_0$  is region occupied by the body in the reference configuration,  $\partial\Omega_0$  its boundary,  $\mathbf{F}$  is the deformation gradient tensor,  $\boldsymbol{\Sigma}$  is the second Piola-Kirchhoff stress tensor,  $\rho_0 f_0$  the reference mass force and  $\mathbf{T}$  the nominal stress vector. Attention should be drawn on the fact that here for an inflatable beam the virtual power  $\int_{\partial\Omega_0} \mathbf{T} \mathbf{V}^* dS_0$  on the beam surface must include the contribution of the internal pressure  $\int_{S_p} \mathbf{V}^* p \mathbf{n} dS$ , computed over the *current* portion of surface  $S_p$ , with outward normal vector  $\mathbf{n}$ , where the pressure is applied (more than often  $S_p$  is the entire internal surface of the membrane).

### 2.1 Kinematics

The Timoshenko's kinematics is assumed for the beam, according to which the cross section remains plane during the deformation but not perpendicular to the bent beam axis. Let  $\mathbf{U}(X, t) = (U(X, t), V(X, t), 0)$  be the displacement of the centroid  $G_0$  of a cross section at reference abscissa  $X$  (all the components are related to basis  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ ), and  $\theta(X, t)$  the rotation about  $\mathbf{z}$  of the section, see Figure 3. The displacement of any particle of reference position  $P_0$  is then given by

$$\mathbf{U}(P_0, t) = \mathbf{U}(G_0, t) + (\mathbf{R} - \mathbf{I}) \cdot \mathbf{G}_0 \mathbf{P}_0$$

$$= \begin{bmatrix} U \\ V \\ 0 \end{bmatrix} + \begin{bmatrix} \cos \theta - 1 & -\sin \theta & 0 \\ \sin \theta & \cos \theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} U - Y \sin \theta \\ V - Y(\cos \theta - 1) \\ 0 \end{bmatrix} \quad (2)$$

where  $\mathbf{R}$  and  $\mathbf{I}$  are the rotation and the identity tensors, respectively.

The components of the Green strain  $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})$  are

$$E_{XX} = U_{,X} - Y \cos \theta \theta_{,X} + \frac{1}{2}(U_{,X}^2 + V_{,X}^2 + Y^2 \theta_{,X}^2) - (U_{,X} \cos \theta \theta_{,X} Y + V_{,X} \sin \theta \theta_{,X} Y)$$

$$E_{YX} = \frac{1}{2}[V_{,X} \cos \theta - (1 + U_{,X}) \sin \theta]$$

$$E_{YY} = 0 \quad (3)$$

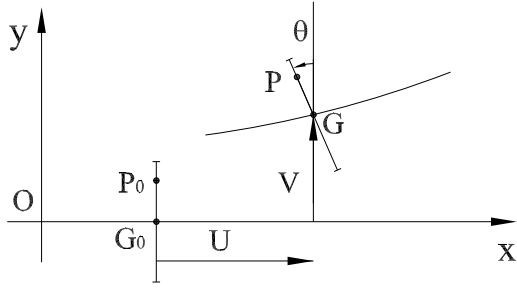


FIGURE 3. KINEMATICS OF THE INFLATABLE BEAM

## 2.2 Virtual kinematics

The virtual velocity field  $\mathbf{V}^*$  in Relation (2) is chosen as follows:

$$\begin{aligned} \mathbf{V}^*(P_0) &= \mathbf{V}^*(G_0) + \Omega^* \times \mathbf{GP} \\ &= \begin{bmatrix} U^* \\ V^* \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \theta^* \end{bmatrix} \times \begin{bmatrix} -Y \sin \theta \\ Y \cos \theta \\ Z \end{bmatrix} = \begin{bmatrix} U^* - Y \cos \theta \theta^* \\ V^* - Y \sin \theta \theta^* \\ 0 \end{bmatrix} \end{aligned} \quad (4)$$

where  $U^*(X), V^*(X)$  are the virtual displacements of the centroid  $G_0$  and  $\theta^*(X)$  the virtual rotation about  $\mathbf{z}$  of the section. There comes the matrix of tensor  $\mathbf{gradV}^*$  in basis  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ :

$$[\mathbf{gradV}^*] = \begin{bmatrix} U_{,X}^* - Y \cos \theta \theta_{,X}^* + Y \sin \theta \theta_{,X}^* \theta^* - \cos \theta \theta^* & 0 \\ V_{,X}^* - Y \sin \theta \theta_{,X}^* - Y \cos \theta \theta_{,X}^* \theta^* - \sin \theta \theta^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5)$$

## 2.3 External virtual power

The virtual power of the external dead loads is well-known. By denoting  $p_X, p_Y$  and  $\mu$  the distributed loads and torque along  $\mathbf{x}, \mathbf{y}$  and  $\mathbf{z}$ , respectively;  $X(\cdot), Y(\cdot)$  and  $\Gamma(\cdot)$  the resultant loads and torques at the ends  $X = 0$  and  $X = \ell_0$ , this virtual power writes

$$\begin{aligned} W_{\text{dead loads}}^* &= \int_{\Omega_0} \rho_0 \mathbf{f}_0 \mathbf{V}^* d\Omega_0 + \int_{\partial\Omega_0} \mathbf{T} \mathbf{V}^* dS_0 \\ &= \int_0^{\ell_0} (p_X U^* + p_Y V^* + \mu \theta^*) dX \\ &\quad + X(0)U^*(0) + Y(0)V^*(0) + \Gamma(0)\theta^*(0) \\ &\quad + X(\ell_0)U^*(\ell_0) + Y(\ell_0)V^*(\ell_0) + \Gamma(\ell_0)\theta^*(\ell_0) \end{aligned} \quad (6)$$

In the above expression, only dead loads are taken into account in the nominal stress  $\mathbf{T}$ . As said previously, one has to

incorporate the virtual power of the internal pressure  $p$ :

$$\begin{aligned} W_{\text{pressure}}^* &= \int_{S_p} \mathbf{V}^* p \mathbf{n} dS = P \int_0^{\ell_0} \{U^* \sin \theta \theta_{,X} - V^* \cos \theta \theta_{,X} \\ &\quad + \theta^* [V_{,X} \cos \theta - (1 + U_{,X}) \sin \theta]\} dX \\ &\quad + P \cdot (U^* \cos \theta + V^* \sin \theta) \Big|_0^{\ell_0} \end{aligned} \quad (7)$$

where  $P = p\pi R_0^2$  the pressure resultant over the reference cross section.

## 2.4 Internal virtual power

The internal virtual power is obtained in a somewhat standard way from Relations (2) and (5):

$$\begin{aligned} \int_{\Omega_0} (\mathbf{F}\Sigma)^T : \mathbf{gradV}^* d\Omega_0 &= \int_0^{\ell_0} \left\{ [N(1 + U_{,X}) + M \cos \theta \theta_{,X} - T \sin \theta] U_{,X}^* \right. \\ &\quad + [NV_{,X} + M \sin \theta \theta_{,X} + T \cos \theta] V_{,X}^* \\ &\quad - [M(1 + U_{,X}) \sin \theta \theta_{,X} + MV_{,X} \cos \theta \theta_{,X} \\ &\quad \left. - [(1 + U_{,X}) \cos \theta + V_{,X} \sin \theta] T] \theta^* \right. \\ &\quad + [M(1 + U_{,X}) \cos \theta \\ &\quad \left. + MV_{,X} \sin \theta + \int_{S_0} Y^2 \Sigma_{XX} dS_0 \theta_{,X} \right] \theta_{,X}^* \} dX \end{aligned} \quad (8)$$

In the above,  $N, M$  and  $T$  are defined as the nominal stress resultants (axial force, bending moment and shear):

$$N \equiv \int_{S_0} \Sigma_{XX} dS_0 \quad M \equiv - \int_{S_0} Y \Sigma_{XX} dS_0 \quad T \equiv \int_{S_0} \Sigma_{XY} dS_0 \quad (9)$$

The particular sum  $\int_{S_0} Y^2 \Sigma_{XX} dS_0$  in (8) will be computed further when the constitutive law is settled.

## 2.5 Strong equations of the problem

Using Relations (6), (7) and (8) leads to the following equilibrium equations for the inflated beam:

$$\begin{aligned} & -[N(1+U_{,X})]_{,X} - (M \cos \theta \theta_{,X})_{,X} + (T \sin \theta)_{,X} - P \sin \theta \theta_{,X} = p_X \\ & -(NV_{,X})_{,X} - (M \sin \theta \theta_{,X})_{,X} - (T \cos \theta)_{,X} + P \cos \theta \theta_{,X} = p_Y \\ & -[M(1+U_{,X})]_{,X} \cos \theta - (MV_{,X})_{,X} \sin \theta - T(1+U_{,X}) \cos \theta \\ & \quad + TV_{,X} \sin \theta - \left( \int_{S_0} Y^2 \Sigma_{XX} dS_0 \theta_{,X} \right)_{,X} \\ & \quad - P[V_{,X} \cos \theta - (1+U_{,X}) \sin \theta] = \mu \end{aligned} \quad (10)$$

together with the boundary conditions:

$$\begin{aligned} & N(0)[1+U_{,X}(0)] + M(0) \cos \theta(0) \theta_{,X}(0) \\ & \quad - T(0) \sin \theta(0) - P \cos \theta(0) = -X(0) \\ & N(\ell_0)[1+U_{,X}(\ell_0)] + M(\ell_0) \cos \theta(\ell_0) \theta_{,X}(\ell_0) \\ & \quad - T(\ell_0) \sin \theta(\ell_0) - P \cos \theta(\ell_0) = +X(\ell_0) \\ & N(0)V_{,X}(0) + M(0) \sin \theta(0) \theta_{,X}(0) \\ & \quad - T(0) \cos \theta(0) - P \sin \theta(0) = -Y(0) \\ & N(\ell_0)V_{,X}(\ell_0) + M(\ell_0) \sin \theta(\ell_0) \theta_{,X}(\ell_0) \\ & \quad - T(\ell_0) \cos \theta(\ell_0) - P \sin \theta(\ell_0) = +Y(\ell_0) \\ & M(0)[1+U_{,X}(0)] \cos \theta(0) + M(0)V_{,X}(0) \sin \theta(0) \\ & \quad + \left( \int_{S_0} Y^2 \Sigma_{XX} dS_0 \right)(0) \theta_{,X}(0) = -\Gamma(0) \\ & M(\ell_0)[1+U_{,X}(\ell_0)] \cos \theta(\ell_0) + M(\ell_0)V_{,X}(\ell_0) \sin \theta(\ell_0) \\ & \quad + \left( \int_{S_0} Y^2 \Sigma_{XX} dS_0 \right)(\ell_0) \theta_{,X}(\ell_0) = +\Gamma(\ell_0) \end{aligned} \quad (11)$$

## 3 ORTHOTROPIC CONSTITUTIVE LAWS

The beam is assumed to be made of a hyperelastic orthotropic fabric, obeying the Saint-Venant Kirchhoff constitutive law:

$$\Sigma = \Sigma^0 + \mathbf{D} : \mathbf{E} \quad (12)$$

where  $\Sigma^0$  is the pre-stress tensor existing in the reference configuration and  $\mathbf{D}$  the fourth-order elasticity tensor. Here,  $\mathbf{D}$  is characterized by Young moduli  $E_\ell$  in the warp direction ( $\ell$ ),  $E_t$  in the weft direction ( $t$ ), in-plane shear modulus  $G_{\ell t}$ , and Poisson's ratios  $\nu_{t\ell}$  and  $\nu_{\ell t}$  (satisfying the equality  $\frac{\nu_{\ell t}}{E_\ell} = \frac{\nu_{t\ell}}{E_t}$ ).

The initial stress  $\Sigma^0$  is induced by the preliminary inflation of the beam, its matrix in the local orthotropy basis ( $\mathbf{e}_\ell, \mathbf{e}_r, \mathbf{e}_t$ )

takes the following form

$$\text{Mat}(\Sigma^0; \mathbf{e}_\ell, \mathbf{e}_r, \mathbf{e}_t) = \begin{bmatrix} \Sigma_{XX}^0 & 0 & \Sigma_{Xt}^0 \\ 0 & 0 & 0 \\ \Sigma_{tX}^0 & 0 & \Sigma_{tt}^0 \end{bmatrix} \quad (13)$$

Its components are related to the Green pre-strain tensor  $\mathbf{E}^0$  by

$$\begin{aligned} \Sigma_{XX}^0 &= \bar{E}_\ell E_{XX}^0 + \nu_{t\ell} \bar{E}_\ell E_{tt}^0 \\ \Sigma_{tt}^0 &= \nu_{\ell t} \bar{E}_t E_{XX}^0 + \bar{E}_t E_{tt}^0 \\ \Sigma_{Xt}^0 &= 2G_{\ell t} E_{Xt}^0 \end{aligned} \quad (14)$$

where  $\bar{E}_\ell = \frac{E_\ell}{1 - \nu_{\ell t} \nu_{t\ell}}$ ,  $\bar{E}_t = \frac{E_t}{1 - \nu_{\ell t} \nu_{t\ell}}$ . The part of stress  $\Sigma^b = \mathbf{D} : \mathbf{E}$  in (12) is due to the bending. It is assumed that its matrix can be written in the local orthotropy basis ( $\mathbf{e}_\ell, \mathbf{e}_r, \mathbf{e}_t$ ) in the following form

$$\text{Mat}(\Sigma^b; \mathbf{e}_\ell, \mathbf{e}_r, \mathbf{e}_t) = \begin{bmatrix} \Sigma_{XX}^b & 0 & \Sigma_{Xt}^b \\ 0 & 0 & 0 \\ \Sigma_{tX}^b & 0 & 0 \end{bmatrix} \quad (15)$$

where the components are

$$\Sigma_{XX}^b = E_\ell \cdot E_{XX} \quad \Sigma_{Xt}^b = 2G_{\ell t} E_{Xt} \quad (16)$$

Thus, the constitutive relationships in the orthotropy basis write

$$\begin{aligned} \Sigma_{XX} &= \Sigma_{XX}^0 + E_\ell \cdot E_{XX} \\ \Sigma_{Xt} &= \Sigma_{Xt}^0 + 2G_{\ell t} E_{Xt} \\ \Sigma_{tt} &= \Sigma_{tt}^0 \end{aligned} \quad (17)$$

The constitutive relationships in the fixed basis ( $\mathbf{xyz}$ ) can be easily derived from (17). The change of basis matrix involves angle  $\varphi$  defined in Figure 2 and shows that the form of the stress matrix is slightly different from that in the orthotropy basis:

$$\begin{aligned} \text{Mat}(\Sigma; \mathbf{xyz}) &= \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} & \Sigma_{XZ} \\ \Sigma_{YX} & \Sigma_{YY} & \Sigma_{YZ} \\ \Sigma_{ZX} & \Sigma_{ZY} & \Sigma_{ZZ} \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_{XX}^0 & \Sigma_{XY}^0 & \Sigma_{XZ}^0 \\ \Sigma_{YX}^0 & \Sigma_{YY}^0 & \Sigma_{YZ}^0 \\ \Sigma_{ZX}^0 & \Sigma_{ZY}^0 & \Sigma_{ZZ}^0 \end{bmatrix} + \begin{bmatrix} \Sigma_{XX}^b & \Sigma_{XY}^b & \Sigma_{XZ}^b \\ \Sigma_{YX}^b & 0 & 0 \\ \Sigma_{ZX}^b & 0 & 0 \end{bmatrix} \end{aligned} \quad (18)$$

We shall retain only two constitutive laws which are useful afterwards, the first is (17)<sub>a</sub> and the second is

$$\Sigma_{XY} = \Sigma_{XY}^0 + 2G_{\ell t}E_{XY} \quad (19)$$

Now, inserting (17)<sub>a</sub> and (19) in definition (9) gives the expressions for the nominal stress resultants in terms of displacements  $U$ ,  $V$  and rotation  $\theta$ :

$$\begin{aligned} N &= N^0 + E_\ell S^0 [U_{,X} + \frac{1}{2}(U_{,X}^2 + V_{,X}^2 + \frac{I_0}{S_0}\theta_{,X}^2)] \\ T &= T^0 + kG_{\ell t}S_0 [V_{,X} \cos \theta - (1 + U_{,X}) \sin \theta] \\ M &= M^0 + E_\ell I_0 [(1 + U_{,X}) \cos \theta + V_{,X} \sin \theta] \theta_{,X} \end{aligned} \quad (20)$$

where  $k$  is the shear factor, equal to  $1/2$  for beams with a circular cross section.

On the other hand, the constitutive law (17)<sub>a</sub> also allows to get an explicit expression for the term  $\int_{S_0} Y^2 \Sigma_{XX} dS_0$  in Equation (8). By assuming that the axial pre-stress is  $\Sigma_{XX}^0 = \alpha^0 + \beta^0 Y + \gamma^0 Y^2$ , which is general enough for our purpose, we find

$$\int_{S_0} Y^2 \Sigma_{XX} dS_0 = \frac{NI_0}{S_0} + \frac{1}{2}EK\theta_{,X}^2 + K\gamma_0 \quad (21)$$

where  $K \equiv \int_{S_0} Y^4 dS_0 - \frac{I_0^2}{S_0}$  is a coefficient which only depends on the cross section geometry.

#### 4 LINEARIZATION

The linearization of the equations will be performed about the pre-stressed reference configuration where the beam is only subjected to the internal pressure  $p$ . To do this, one needs to make the following assumptions which are usually met in practice:

- About the magnitudes of the kinematic variables:
  - .  $V/\ell_0$  and  $\theta$  are infinitesimal of order 1,
  - .  $U/\ell_0$  is infinitesimal of order 2.
- About the initial stresses in (20) and (21):
  - .  $\Sigma_{XX}^0 = \text{constant} \Rightarrow M^0 = 0, \gamma^0 = 0,$
  - .  $\Sigma_{XY}^0 = 0 \Rightarrow T^0 = 0.$

Taking these assumptions into account in (20) gives the linearized expressions for the the nominal stress resultants:

$$N = N^0 \quad M = E_\ell I_0 \theta_{,X} \quad T = kG_{\ell t}S_0 (V_{,X} - \theta) \quad (22)$$

Equations (10) then yield the linearized equilibrium equations:

$$\begin{aligned} -N_{,X}^0 &= p_X \\ -(N^0 + kG_{\ell t}S_0)V_{,X} + (P + kG_{\ell t}S_0)\theta_{,X} &= p_Y \\ -(E_\ell + \frac{N^0}{I_0})I_0\theta_{,X} - (P + kG_{\ell t}S_0)(V_{,X} - \theta) &= \mu \end{aligned} \quad (23)$$

whereas Relations (11) give the linearized boundary conditions:

$$\begin{aligned} N^0(0) - P &= -X(0) \\ N^0(\ell_0) - P &= +X(\ell_0) \\ (N^0(0) + kG_{\ell t}S_0)V_{,X}(0) - (P + kG_{\ell t}S_0)\theta(0) &= -Y(0) \\ (N^0(\ell_0) + kG_{\ell t}S_0)V_{,X}(\ell_0) - (P + kG_{\ell t}S_0)\theta(\ell_0) &= +Y(\ell_0) \\ (E_\ell + \frac{N^0(0)}{I_0})I_0\theta_{,X}(0) &= -\Gamma(0) \\ (E_\ell + \frac{N^0(\ell_0)}{I_0})I_0\theta_{,X}(\ell_0) &= +\Gamma(\ell_0) \end{aligned} \quad (24)$$

#### 5 RADIUS AND LENGTH IN THE REFERENCE STATE

When the internal pressure is low, radius  $R_0$  and length  $\ell_0$  in the reference state can be obtained using small deformation assumptions. In the cylindrical part of the beam, the axial and loop Cauchy stresses are  $\sigma_{\ell\ell} = \frac{pR_\phi}{2h_\phi}$  and  $\sigma_{tt} = \frac{pR_\phi}{h_\phi}$ ; and the axial and loop strains are  $\varepsilon_{\ell\ell} = \frac{\ell_0 - \ell_\phi}{\ell_\phi}$  and  $\varepsilon_{tt} = \frac{R_0 - R_\phi}{R_\phi}$ . By substituting these equalities into the constitutive relations

$$\begin{aligned} \sigma_{\ell\ell} &= \bar{E}_\ell \varepsilon_{\ell\ell} + \nu_{\ell t} \bar{E}_t \varepsilon_{tt} \\ \sigma_{tt} &= \nu_{\ell t} \bar{E}_t \varepsilon_{\ell\ell} + \bar{E}_t \varepsilon_{tt} \end{aligned} \quad (25)$$

, one obtains:

$$\begin{aligned} \ell_0 &= \ell_\phi + \frac{pR_\phi \ell_\phi}{2E_\ell h_\phi} (1 - 2\nu_{\ell t}) \\ R_0 &= R_\phi + \frac{pR_\phi^2}{2E_t h_\phi} (2 - \nu_{\ell t}) \end{aligned} \quad (26)$$

These relations are simple and are valid in the linear theory. However, when the pressure is high, the previous relations are no longer valid and nonlinear relationships should be used in order to derive accurate radius and length, as will be shown next. One has to consider the deformation gradient tensor:

$$Mat(\mathbf{F}; \mathbf{e}_\ell \mathbf{e}_r \mathbf{e}_t) = \begin{bmatrix} \frac{\ell_0}{\ell_\phi} & 0 & 0 \\ 0 & \frac{h_0}{h_\phi} & 0 \\ 0 & 0 & \frac{R_0}{R_\phi} \end{bmatrix} \quad (27)$$

and the jacobian is  $J = \frac{R_0 \ell_0 h_0}{R_\varnothing \ell_\varnothing h_\varnothing}$ . The second Piola-Kirchoff stress tensor is obtained from  $\Sigma = J \mathbf{F}^{-1} \boldsymbol{\sigma} \mathbf{F}^{-T}$ :

$$\text{Mat}(\Sigma; \mathbf{e}_\ell \mathbf{e}_r \mathbf{e}_t) = \begin{bmatrix} \frac{p R_0^2}{2 R_\varnothing h_\varnothing} \frac{\ell_\varnothing}{\ell_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{p R_\varnothing}{h_\varnothing} \frac{\ell_0}{\ell_\varnothing} \end{bmatrix} \quad (28)$$

From (27), the Green strain tensor writes:

$$\text{Mat}(\mathbf{E}; \mathbf{e}_\ell \mathbf{e}_r \mathbf{e}_t) = \frac{1}{2} \begin{bmatrix} \left(\frac{\ell_0}{\ell_\varnothing}\right)^2 - 1 & 0 & 0 \\ 0 & \left(\frac{h_0}{h_\varnothing}\right)^2 - 1 & 0 \\ 0 & 0 & \left(\frac{R_0}{R_\varnothing}\right)^2 - 1 \end{bmatrix} \quad (29)$$

Inserting (28)-(29) into the constitutive relationship

$$\begin{aligned} \Sigma_{XX}^0 - \nu_{t\ell} \Sigma_{YY}^0 &= E_\ell E_{XX}^0 \\ \Sigma_{YY}^0 - \nu_{t\ell} \Sigma_{XX}^0 &= E_t E_{YY}^0 \end{aligned} \quad (30)$$

one gets the following set of equations for the reference radius  $R_0$  and length  $\ell_0$ :

$$\begin{aligned} \frac{p R_0^2}{2 R_\varnothing} \frac{\ell_\varnothing}{\ell_0} - \nu_{t\ell} p R_\varnothing \frac{\ell_0}{\ell_\varnothing} &= h_\varnothing E_\ell \frac{1}{2} \left[ \left(\frac{\ell_0}{\ell_\varnothing}\right)^2 - 1 \right] \\ p R_\varnothing \frac{\ell_0}{\ell_\varnothing} - \nu_{t\ell} \frac{p R_0^2}{2 R_\varnothing} \frac{\ell_\varnothing}{\ell_0} &= h_\varnothing E_t \frac{1}{2} \left[ \left(\frac{R_0}{R_\varnothing}\right)^2 - 1 \right] \end{aligned} \quad (31)$$

Equation (31)<sub>a</sub> can be used to eliminate radius  $R_0$ :

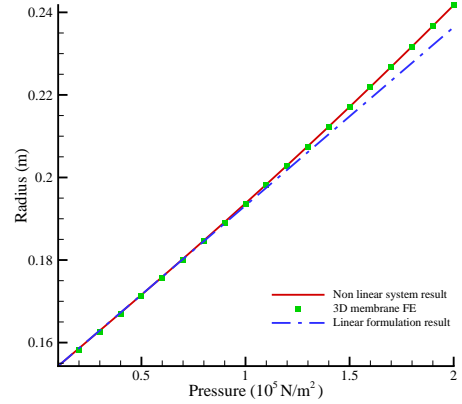
$$\boxed{\frac{R_0}{R_\varnothing} = \frac{h_\varnothing E_\ell}{p R_\varnothing} L^3 + 2 \nu_{t\ell} L^2 - \frac{h_\varnothing E_\ell}{p R_\varnothing} L} \quad (L = \frac{\ell_0}{\ell_\varnothing}) \quad (32)$$

Hence, one obtains a cubic equation for  $L = \frac{\ell_0}{\ell_\varnothing}$ :

$$\boxed{\frac{h_\varnothing E_\ell}{p R_\varnothing} L^3 + 3 \nu_{t\ell} L^2 - \left( \frac{h_\varnothing E_\ell}{p R_\varnothing} + 2 \frac{p R_\varnothing}{h_\varnothing E_t} (1 - \nu_{t\ell} \nu_{t\ell}) \right) L - (1 + \nu_{t\ell}) = 0} \quad (33)$$

which can be solved by means of Cardan's formula to give an analytical expression for the reference length  $\ell_0$ .

As an example, let us solve Equations (32)-(33) with the following material properties of the fabric  $h_\varnothing E_\ell = h_\varnothing E_t = 50000 \text{ Pa.m}$ ,  $h_\varnothing G_{t\ell} = 12500 \text{ Pa.m}$  and  $\nu_{t\ell} = 0.08$ . The theoretical radius and length obtained are shown in Figures 4 and 5, together with results from linear formulation (26) and numerical values from a nonlinear finite element code dedicated to 3D orthotropic membranes with zero bending stiffness and satisfying the plane stress conditions. Clearly, the reference radius  $R_0$  and more importantly the reference length  $\ell_0$  are nonlinear functions of the internal pressure  $p$ . In the case at hand, the difference between the linear and nonlinear formulations is greater than 10% for the length and about 2% for the radius. On the other hand, the results of the nonlinear equations (32)-(33) agree very well with those obtained from the 3D membrane code.



**FIGURE 4.** REFERENCE RADIUS OF THE INFLATED TUBE VS. THE INTERNAL PRESSURE

## 6 APPLICATION TO AN INFLATED CANTILEVER BEAM

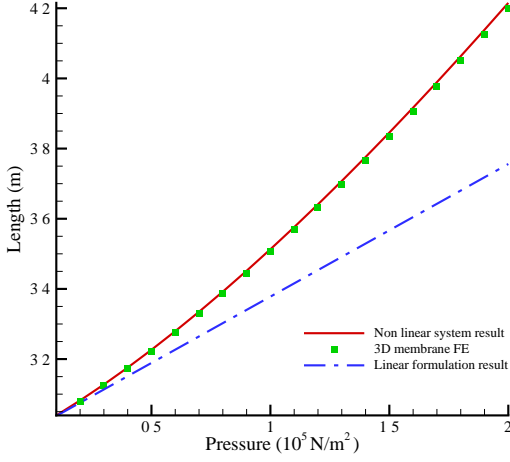
The above results are now applied to the bending problem of a cantilever orthotropic beam subjected to a concentrated load at the free end. The beam is made of an orthotropic membrane, with reference length  $\ell_0$ , radius  $R_0$  and thickness  $h_0$ . It is clamped at end  $X = 0$ , subjected to an internal pressure  $p$  and a transverse force  $F$  at end  $X = \ell_0$ , see Figure (6).

### 6.1 Deflection and rotation

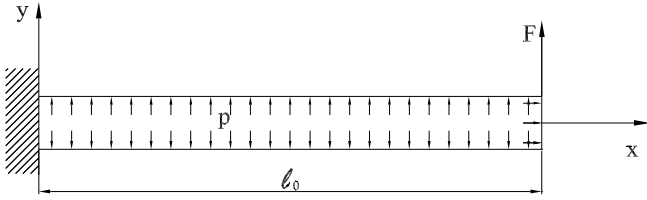
Equations (23)<sub>a</sub> and (24) readily yield the initial axial force  $N^0(X) = P$  and the static boundary conditions

$$\theta_{,X}(\ell_0) = 0 \quad (P + k G_{t\ell} S_0)(V_{,X}(\ell_0) - \theta(\ell_0)) = F \quad (34)$$





**FIGURE 5.** REFERENCE LENGTH OF THE INFLATED TUBE VS. THE INTERNAL PRESSURE



**FIGURE 6.** INFLATED CANTILEVER UNDER BENDING LOAD

Solving (23)<sub>b-c</sub> and (34) with the kinematic boundary conditions  $V(0) = \theta(0) = 0$  leads to the deflection and rotation of the beam:

$$\begin{aligned} V(X) &= \frac{F}{(E_\ell + \frac{P}{S_0})I_0} \left( \frac{\ell_0 X^2}{2} - \frac{X^3}{6} \right) + \frac{FX}{P + kG_{\ell t}S_0} \\ \theta(X) &= \frac{F}{(E_\ell + \frac{P}{S_0})I_0} \left( \ell_0 X - \frac{X^2}{2} \right) \end{aligned} \quad (35)$$

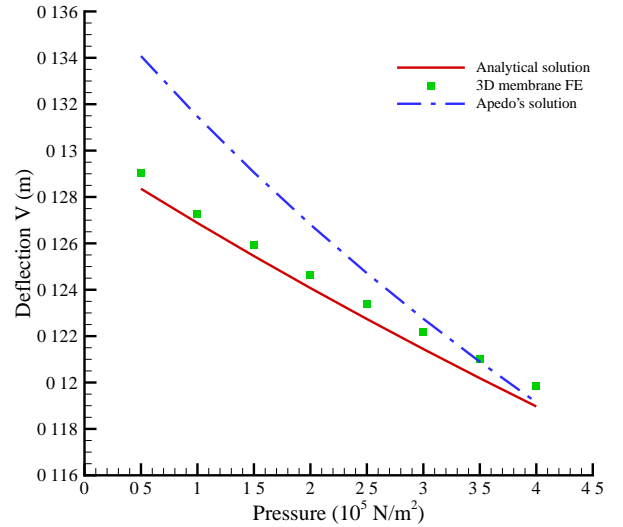
These relations clearly show the important role of the internal pressure  $p$  – via the pressure resultant  $P = p\pi R_0^2$  – in the stiffness of the beam. Equations (35) are almost the same as in [8], but  $\bar{E}_\ell = \frac{E_\ell}{1 - \nu_{\ell t}\nu_{t\ell}}$  therein is replaced by  $E_\ell$  and  $\frac{1}{2}kG_{\ell t}S_0$  by  $kG_{\ell t}S_0$ .

Figures 7 and 8 show the deflection and rotation at the free end  $X = \ell_0$ , using the numerical data in Table 1. As can be seen,

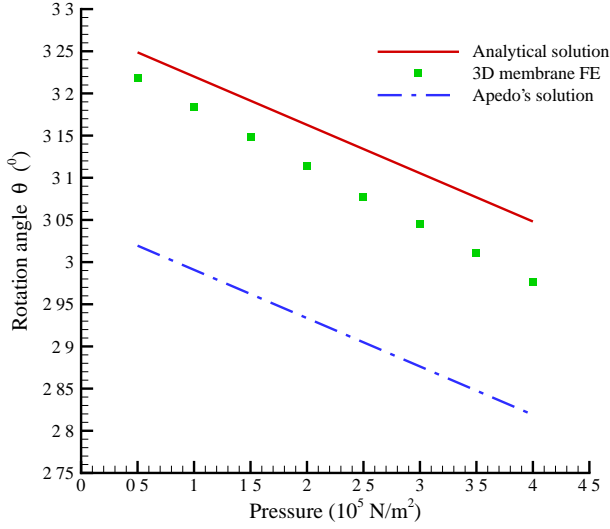
**TABLE 1.** GEOMETRY AND MATERIAL PROPERTIES OF THE CANTILEVER BEAM

Natural length $\ell_\circ$	3 m
Natural radius $R_\circ$	0.2 m
Natural thickness $h_\circ$	$0.125 \cdot 10^{-3}$ m
Warp Young's modulus $E_\ell$	2500 MPa
Weft Young's Modulus $E_t$	2000 MPa
In-plane shear modulus $G_{\ell t}$	250 MPa
Poisson ratios $\nu_{\ell t}$ and $\nu_{t\ell}$	0.3    0.24
Internal pressure $p$ ( $10^5$ Pa)	0.5    1    1.5    2 2.5    3    3.5    4
Concentrate load F	1 kN

the analytical solutions (35) yield results which are in very good accordance with the 3D membrane computations, the maximum error is 0.6% for the deflection and 1.1% for the rotation, thus improving the solution given in [8].



**FIGURE 7.** DEFLECTION AT THE TIP OF AN INFLATED CANTILEVER BEAM VS. THE INTERNAL PRESSURE



**FIGURE 8.** ROTATION AT THE TIP OF AN INFLATED CANTILEVER BEAM VS. THE INTERNAL PRESSURE

## 6.2 Wrinkling load

It should be noticed that, given a geometry, material properties and an internal pressure, there exists a critical load  $F_c$ , referred to as the wrinkling load, above which wrinkles may appear in the membrane and the proposed model is no longer valid. The wrinkling load can be determined using the conditions that the principal stresses at any point in the membrane are non-negative. Here, it can be shown that the wrinkling load depends on the beam geometry and the internal pressure as:

$$F_c = \frac{\pi R_0^3 p}{2I_0} \quad (36)$$

This relation is the same as in [7] which was found for isotropic fabric. Note that the value  $F = 1$  kN chosen in the numerical computation above is lower than  $F_c$ .

## 7 CONCLUSIONS

This paper has been devoted to the theoretical analysis of inflatable beam made of an orthotropic fabric by means of the total Lagrangian formulation. The following results have been obtained:

(i) The nonlinear equations set which clearly show how the internal pressure – an essential ingredient in inflatable beams – is involved.

(ii) The linearized equations about the pre-stressed reference configuration, which enable one to derive analytical deflections and rotations under prescribed boundary conditions and external loads.

(iii) The geometry of the pressurized beam in the pre-stressed reference configuration: it can be determined by solving a cubic equation.

The solution of these equations has been carried out in the case of a cantilever beam has shown the results agree very well with 3D finite element membrane computations.

It was assumed in the proposed formulation that the orthotropic axes of the fabric coincide with the cylindrical local axes of the beam. Further studies without this assumption are under progress.

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